TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 227, 1977

ULM'S THEOREM FOR PARTIALLY ORDERED STRUCTURES RELATED TO SIMPLY PRESENTED ABELIAN p-GROUPS

BY

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ABSTRACT. If we have an abelian p-group G, a multiplication by p for each element of G is defined by setting $px = x + x + \cdots + x$, where p is the number of terms in the sum. If we forget about the addition on G, and just keep the multiplication by p, we have the algebraic structure called a p-basic tree. A natural partial order can be defined, the graph of which is a tree with 0 as root. A p-basic tree generates a simply presented abelian p-group, and provides a natural direct sum decomposition for it. Ulm invariants may be defined directly for a p-basic tree so that they are equal to the Ulm invariants of the corresponding group. A central notion is that of a stripping function between two p-basic trees. Given a stripping function from X onto Y we can construct an isomorphism between the groups corresponding to X and Y; in particular, X and Y have the same Ulm invariants. Conversely, if X and Y have the same Ulm invariants, then there is a map from X onto Y that is the composition of two stripping functions and two inverses of stripping functions. These results constitute Ulm's theorem for p-basic trees, and provide a new proof of Ulm's theorem for simply presented groups.

1. **Introduction.** In 1933, Ulm [7] characterized countable reduced abelian p-groups in terms of functions from ordinals to cardinals defined by $f(\alpha, G) = \dim p^{\alpha}G[p]/p^{\alpha+1}G[p]$, the Ulm invariants of G. This result was extended by Kolettis [6] in 1960 to direct sums of such groups, and by Hill [4], [5] in 1967 to the class of all totally projective groups. In 1969, Crawley and Hales [1] proved Ulm's theorem for reduced simply presented p-groups (T-groups, in their terminology), and showed that the class of reduced simply presented p-groups coincides with the class of all totally projective groups.

The principal object of study in this paper is the p-basic tree, which is a set upon which multiplication by a prime p is defined. A p-basic tree generates a simply presented p-group, and provides a natural direct sum decomposition for it. Conversely, any such group contains a p-basic tree which generates it. Ulm invariants can be defined directly for a p-basic tree so that they are equal to the Ulm invariants of the corresponding group.

A central notion introduced here is that of a stripping function between two p-basic trees. Given a stripping function, we can construct an isomorphism between the corresponding groups (Theorem 2). If a map between two

Received by the editors January 5, 1976.
AMS (MOS) subject classifications (1970). Primary 20K10; Secondary 06A10.

p-basic trees exists which is the composition of a stripping function and the inverse of a stripping function (call such a map a *strip-graft*), then the natural direct sum decompositions of the corresponding groups have isomorphic refinements (Theorem 3). A principal result (Theorem 4) is that two p-basic trees generate isomorphic groups if and only if a map between them exists which is the composition of two strip-grafts. We finish with a version of Ulm's theorem for p-basic trees (Theorem 5), which gives a new proof of Ulm's theorem for simply presented groups.

2. **Preliminaries.** Throughout the discussion, the letter p shall denote a fixed but arbitrary prime, and "group" shall mean "abelian p-group". We shall conform to standard terminology and notation as used, for example, in [3].

If we have a group G, a multiplication by p for each element of G is defined by setting $px = x + x + \cdots + x$, where p is the number of terms in the sum. This is a most important structure on G; in particular, it has the property that given g in G, there is an integer n with $p^n g = 0$. If we forget about the addition on G, and just keep the multiplication by p, we have the algebraic structure which we shall call a p-basic tree.

DEFINITION 1. A *p-basic tree* is a set X together with a multiplication by p for elements of X such that:

(1) $px \in X$ for each x in X,

(2) there is an element 0 in X for which $p0 = 0$, and

(3) for each nonzero element x in X, there is a positive integer n such that $p^n x = 0$.

If $x \in X$, the least integer n such that $p^n x = 0$ is the *exponent* of x. The relation $>$ on X is defined as follows: for elements x, y of X, if there is a positive integer m such that $p^m x = y$, then $x > y$. Clearly, the relation $\geqslant$ is a partial order. The graph of X under this partial order is a tree with 0 as root. An *atom* of X is a nonzero element z with the property that, if $z \geqslant x$, then $x = z$ or $x = 0$. For each x in X, set $C(x) = \{y \in X : y \geqslant x \text{ or } y < x\}$. A subset S of X is a *subtree* of X if, for each s in S, the element ps is in S. A collection $\{S_i\}$ of subtrees of X is *disjoint* if $S_i \neq S_j$ implies $S_i \cap S_j = \{0\}$. If $\{S_i\}$ is a disjoint collection of subtrees of X such that $X = \bigcup S_i$, then X is the *direct sum* of its subtrees S_i and we write $X = \bigoplus S_i$. Note that if $x \in X$, then $C(x)$ is a subtree, and if A is the set of atoms of X, then $X = \bigoplus_{a \in A} C(a)$. The group whose generators are elements of X, subject to the relations $px = y$, is *the group generated by* X, and shall be denoted by $[X]$. The next definition is taken from [3].

DEFINITION 2. A group G is *simply presented* if it has a set X of generators and a set of relations such that:

(1) for every x in X, $x \neq 0$ in G,

(2) if x, y are distinct elements in X, then $x \neq y$ in G, and

(3) all relations are of the form $px = 0$ or $px = y$ where $x, y \in X$.

The set X is a *p-basis* of G. A p-basis can also be characterized in the following way [2], [8]: a subset X of G such that if $x \in X$ and $px \neq 0$, then $px \in X$, and every element in G can be written uniquely in the form $\Sigma_{x \in X} n_x x$ with $0 \leqslant n_x < p$. It is shown in [2] that a group is simply presented iff it has a p-basis. The representation of an element in the form $\Sigma n_x x$ will be referred to as the unique representation of the element as a *linear combination* of elements of X. It is clear that if X is a p-basic tree, then $[X]$ is a simply presented group with p-basis $X \sim \{0\}$; conversely, if a group is simply presented with a p-basis X, then $X \cup \{0\}$ is a p-basic tree.

Observe now that if X is a p-basic tree and $\{S_i\}$ a collection of subtrees such that $X = \bigoplus S_i$, then $[X] = \bigoplus [S_i]$. Consequently, if A is the set of atoms of X, the decomposition $[X] = \bigoplus_{a \in A}[C(a)]$ shall be referred to as the *natural* direct sum decomposition of $[X]$ given by X.

DEFINITION 3. Let X be a p-basic tree. Set $p^0X = X$, and for any ordinal β, set $p^{\beta}X = \bigcap_{\alpha<\beta} p(p^{\alpha}X)$. An element x of X has *height* α if $x \in p^{\alpha}X \sim p^{\alpha+1}X$; we shall denote the height of x by hx. If $x \in p^{\alpha}X$ for every α, then we write $hx = \infty$, where ∞ is considered to be larger than any ordinal. There must be some ordinal λ for which $p^{\lambda}X = p^{\lambda+1}X$; the least such λ is the *length* of X. If $p^{\lambda}X = 0$, then X is *reduced*. Henceforth we shall assume that *all p-basic trees are reduced*. A useful property [1] of simply presented groups is that if $b = \Sigma_{i=1}^{k} r_i x_i$ is the unique representation of a nonzero element b as a linear combination of elements of X, then $b \in p^{\alpha}G$ iff $x_i \in p^{\alpha}G$ for $i = 1, \ldots, k$. This can be strengthened.

PROPOSITION 1. *Suppose X is a p-basic tree, and b is a nonzero element of $[X]$ whose representation as a linear combination of elements of X is $b = \Sigma_{i=1}^{k} r_i x_i$. If hx_i represents the height in X of x_i, then the height of b in the group $[X]$ is* $\min\{hx_1, \ldots, hx_k\}$.

PROOF. It suffices to verify that $p^{\alpha}X = X \cap p^{\alpha}[X]$ for any ordinal α; this is accomplished by an induction on α. □

DEFINITION 4. Two p-basic trees X and Y are *isomorphic* if there is a height-preserving bijection η: $X \to Y$ such that $\eta(px) = p\eta(x)$ for every x in X; the map η is an *isomorphism*. If the groups $[X]$ and $[Y]$ are isomorphic, then X and Y are *equivalent*. Obviously, isomorphic p-basic trees are equivalent.

3. Ulm invariants and stripping functions. Recall that, for a group G and an ordinal α, the αth Ulm invariant of G is the cardinal number

$$f(\alpha, G) = \dim \; p^{\alpha}G[p]/p^{\alpha+1}G[p].$$

The corresponding notion for p-basic trees is the following.

DEFINITION 5. Let X be a p-basic tree of length λ, and α an ordinal such that $\alpha < \lambda$. Define $U(\alpha, X) = \{x \in X\colon\ hx = \alpha \text{ and } h(px) > \alpha + 1\}$. Denote by A the set of elements of X with height $\alpha + 1$, and suppose $z \in A$. Choose an element in $p^{-1}z$ with height α and denote it by z'. Set $D(z) = \{y \in p^{-1}z\colon hy = \alpha \text{ and } y \neq z'\}$. We now define the αth *Ulm invariant* of X to be the cardinal number

$$f(\alpha, X) = |U(\alpha, X)| + \left| \bigcup_{z \in A} D(z) \right|.$$

Proposition 2. *Let X be a p-basic tree of length λ, and α an ordinal such that $\alpha < \lambda$. Then $f(\alpha, X) = f(\alpha, [X])$.*

Proof. Denote the group $p^{\alpha}[X][p]$ by G, and its subgroup $p^{\alpha+1}[X][p]$ by H. For each x in $U(\alpha, X)$, there is an element u in $p^{\alpha}X$ with $p^2u = px$, so $x - pu + H$ is a nonzero element of G/H. Let $B_1 = \{x - pu + H\colon x \in U(\alpha, X), u \in p^{\alpha}X, \text{ and } p^2u = px\}$. For z in A (where A is the set of elements of height $\alpha + 1$), and y in $D(z)$, the element $y - z' + H$ is a nonzero element of G/H. Let $B_2 = \{y - z' + H\colon z \in A, y \in D(z)\}$. We proceed to show that $B_1 \cup B_2$ is a basis of G/H, considered as a vector space over $Z(p)$.

To see that $B_1 \cup B_2$ is independent, suppose

$$\sum_{i=1}^{n} r_i(x_i - pu_i + H) + \sum_{i=1}^{k} t_{i1}(y_{i1} - z_1' + H) + \cdots + \sum_{i=1}^{k'} t_{im}(y_{im} - z_m' + H) = 0$$

in G/H, where each r_i and each t_{ij} is a nonnegative integer less than p. This implies that in $[X]$, the element

$$\sum_{i=1}^{n} r_ix_i + \sum_{i=1}^{k} t_{i1}y_{i1} + \cdots + \sum_{i=1}^{k'} t_{im}y_{im} - \left(\sum_{i=1}^{k} t_{i1}\right)z_1' - \cdots - \left(\sum_{i=1}^{k'} t_{im}\right)z_m'$$

has height at least $\alpha + 1$. Since all elements from X in this expression are distinct, each $r_i = t_{ij} = 0$.

To see that $B_1 \cup B_2$ spans G/H, suppose $v + H$ is a nonzero element of G/H, and $v = \sum_{i=1}^{n} r_ix_i$ is its representation in $[X]$ as a linear combination of elements of X. Assume that any element in G whose representation contains less than n terms is in the linear span S of $B_1 \cup B_2$ in G/H. In the expression for v, if any $px_i = 0$, then $v - r_ix_i$ satisfies the induction hypothesis, and $x_i + H$ is equal to an element of B_1, so $v + H \in S$. If no $px_i = 0$, then in the set $\{x_1, \ldots, x_n\}$ there must be at least two elements x_i and x_j for which $px_i = px_j$; say $px_1 = \cdots = px_k = z$. If $hz > \alpha + 1$, there is u in $p^{\alpha}X$ such that $p^2u = z$, and so

$$v = r_1(x_1 - pu) + \cdots + r_k(x_k - pu) + \sum_{i=1}^{k} r_i(pu) + \sum_{i=k+1}^{n} r_ix_i.$$

Then $v - \sum_{i=1}^{k} r_i(x_i - pu)$ satisfies the induction hypothesis, and each $x_i - pu$

$+ H$ is equal to an element of B_1, so $v + H \in S$. If $hz = \alpha + 1$, write

$$v = r_1(x_1 - z') + \cdots + r_k(x_k - z') + \left(\sum_{i=1}^{k} r_i\right)z' + \sum_{i=k+1}^{n} r_i x_i.$$

Since $x_i - z' + H \in B_2$, a similar argument shows that $v + H \in S$. □

DEFINITION 6. A p-basic tree is *fully stripped* if, whenever there are distinct elements x and y with $px = py$, then $h(px)$ is a limit ordinal or ∞.

DEFINITION 7. Let X and Y be p-basic trees and $\sigma: X \to Y$ a bijection. Then σ is a *stripping function* if

(1) σ preserves heights, and

(2) if $\sigma(px) \neq p\sigma(x)$, then $p\sigma(x) = 0$. A bijection ξ is a *strip-graft* if there is a p-basic tree Z, a stripping function $\sigma: X \to Z$, and a stripping function $\psi: Y \to Z$ such that $\xi = \psi^{-1}\sigma$. An easy consequence of the definition is the following.

PROPOSITION 3. *Suppose X and Y are p-basic trees and $\sigma: X \to Y$ is a stripping function. Let a be an atom of X, and set $N(a) = \{x \in C(a): p\sigma(x) \neq \sigma(px)\}$. Then σa is an atom of Y, and σx is an atom of Y for each x in $N(a)$. Furthermore, the image of the subtree $C(a)$ is*

$$\sigma[C(a)] = \bigoplus_{x \in N(a)} C(\sigma x) \oplus C(\sigma a).$$

DEFINITION 8. Let X be a p-basic tree and z a nonzero element of X. A subset D of $p^{-1}z$ is *dense* in $p^{-1}z$ if, given an ordinal $\alpha < hz$, there is an element y in D with $\alpha \leq hy$.

LEMMA 1. *Let X be a p-basic tree, and σ a bijection of X onto a set Y. Suppose multiplication by p is defined on Y so that either $p\sigma(x) = \sigma(px)$, or $p\sigma(x) = \sigma(0)$. Then Y is a p-basic tree, and σ is a stripping function iff for every nonzero element z in X, the set $\{y \in p^{-1}z: p\sigma(y) = \sigma(py)\}$ is dense in $p^{-1}z$.*

PROPOSITION 4. *Let X be a p-basic tree. Then there is a fully stripped p-basic tree X' and a stripping function $\sigma: X \to X'$.*

PROOF. Define $*$, a new multiplication by p for X, as follows: $p * x = px$, if $h(px)$ is a limit ordinal or if there is some z in X for which $x = z'$, and $p * x = 0$ otherwise. With this multiplication, X is a fully stripped p-basic tree, which shall be denoted X'. Lemma 1 shows that the identity function σ is a stripping function. □

THEOREM 1. *If X and Y are p-basic trees and $\sigma: X \to Y$ is a stripping function, then X and Y have the same Ulm invariants.*

PROOF. Let α denote an arbitrary ordinal, and A the set of elements of X with height $\alpha + 1$. The set A' of elements of Y with height $\alpha + 1$ coincides with $\sigma[A]$. The result follows from the fact that there is a one-to-one correspondence between the sets $U(\alpha, X) \cup \bigcup_{z\in A} D(z)$ and $U(\alpha, Y) \cup \bigcup_{z\in A} D(\sigma z)$. □

In view of Theorem 1, it is a consequence of Ulm's theorem for simply presented groups that $[X]$ and $[Y]$ are isomorphic. However, we can prove this result directly in a more general setting, and use it to prove Ulm's theorem.

THEOREM 2. *Let X and Y be p-basic trees and suppose σ: $X \to Y$ is a stripping function. Then X and Y are equivalent.*

PROOF. Define the function d on X as follows: set $d(0) = 0$, and if the exponent of x is $k + 1$, set $d(x) = d(px)$ if $p\sigma(x) = \sigma(px)$, and set $d(x) = d(px) + 1$ if $p\sigma(x) \neq \sigma(px)$. We use d to define inductively a function π: $X \to X$ such that for each x in X we have $p\pi(x) = \pi(px)$ iff $p\sigma x = \sigma(px)$, and such that: (1) $h(\pi x) \geqslant hx$, (2) there is a nonnegative integer n for which $\pi^n(x) = 0$, (3) either $p\pi x = \pi px$ or $p\pi x = px$, and (4) if $\pi x \neq 0$, then $d(\pi x) < d(x)$. The construction of π is as follows. If $d(x) = 0$ or if $px = 0$, set $\pi x = 0$. Assume π has been defined to satisfy (1)–(4) for all y such that $d(y) < k$, and for all y such that $d(y) = k$ and $p^{m-1}y = 0$, where k and m are positive integers. Suppose x is an element of X whose exponent is m, and for which $d(x) = k$. If $p\sigma x \neq \sigma px$, then by Lemma 1 there is an element y of X with $py = px$, $hy \geqslant hx$, and $\sigma py = p\sigma y$. Since $d(y) = d(py) = d(px) < d(x)$, πy has been defined. Set $\pi x = y$. If $p\sigma x = \sigma px$, then $d(x) = d(px)$, but the exponent of px is $m - 1$; hence $\pi(px)$ has been defined. If $\pi(px) = 0$ set $\pi x = 0$. Otherwise, note $h(\pi(px)) \geqslant h(px) > hx$, so there is an element u of X with $pu = \pi(px)$ and $hu \geqslant hx$. If $p\sigma u = \sigma pu$, then πu has been defined; set $\pi x = u$. On the other hand, if $p\sigma u \neq \sigma pu$, then by Lemma 1 there is an element w of X with $pw = pu$, $hw \geqslant hu$, and $\sigma(pw) = p\sigma w$. Set $\pi x = w$.

We use π to define yet another function π': $X \to [X]$ by setting $\pi'(x) = x - \pi x$. Denote the image of X in $[X]$ by X'. Then X' is a p-basic tree, and π': $X \to X'$ is a stripping function. Furthermore, $X' \sim \{0\}$ is a p-basis of $[X]$ (to see this, note if $g = \sum_{i=1}^{k} r_i x_i$ in $[X]$, with each x_i in X, a representation of g as a linear combination of elements of X' is

$$g = \sum_{i=1}^{k} \left[r_i(x_i - \pi x_i) + \cdots + r_i\left(\pi^{m-1}x_i - \pi^m x_i\right) \right],$$

where m is large enough so that $\pi^m x_i = 0$ for all $i = 1, \ldots, k$).

Finally, observe that Y and X' are isomorphic under the map η: $Y \to X'$ defined by $\eta(\sigma x) = \pi'(x)$. □

THEOREM 3. *Let X and Y be p-basic trees and ξ: $X \to Y$ a strip-graft. Then the natural direct sum decompositions of $[X]$ and $[Y]$ have isomorphic refinements.*

PROOF. Let Z be the p-basic tree with σ: $X \to Z$ and ψ: $Y \to Z$ the stripping functions such that $\xi = \psi^{-1}\sigma$. Then the set T of atoms of Z is $T = \{\sigma a\colon a \in A\} \cup \{\sigma x\colon p\sigma x \neq \sigma p x\}$, where A is the set of atoms of X. Consequently, $[Z] = \bigoplus_{t \in T}[C(t)]$ is isomorphic to a refinement of $[X] = \bigoplus_{a \in A}[C(a)]$. Similarly, $[Z]$ is isomorphic to a refinement of $[Y]$.

4. p-basic trees with the same Ulm invariants. In what follows, we shall be interested in p-basic trees with the same Ulm invariants, and because of Theorem 1 and Proposition 4, we may restrict our attention to fully stripped p-basic trees.

DEFINITION 9. Suppose X and Y are p-basic trees and θ: $X \to Y$ is a height-preserving bijection which has the property that $\theta(px) = p\theta x$, except possibly when both $h(px)$ and $h(p\theta x)$ are limit ordinals or ∞. Then we shall say that θ is a *T-function*.

Observe that the composition of two T-functions is a T-function; the inverse of a T-function is a T-function; and if both X and Y are fully stripped, and σ: $X \to Y$ is a stripping function, then both σ and σ^{-1} are T-functions.

For the purposes of the next several definitions and propositions, we shall assume that *X and Y are fully stripped p-basic trees*, and *the symbol θ shall be reserved for a T-function.*

PROPOSITION 5. *The p-basic trees X and Y have the same Ulm invariants iff there is a T-function θ: $X \to Y$.*

PROOF. Suppose that X and Y have the same Ulm invariants; this is equivalent to the possession of bijections η_α: $U(\alpha, X) \to U(\alpha, Y)$ for each ordinal α. Note if x is a nonzero element of X with height α and $x \notin U(\alpha, X)$, then there is a least positive integer n such that $p^n x \in U(\zeta, X)$ for the ordinal $\zeta = \alpha + n$. Thus in Y, there is an element y such that $p^n y = \eta(p^n x)$; since Y is fully stripped, y is unique. The T-function θ is defined as follows: set $\theta(0) = 0$, set $\theta(x) = \eta_\alpha(x)$ if $x \in U(\alpha, X)$, and set $\theta(x) = Y$ if $x \notin U(\alpha, X)$ and y is as described above. The converse is easily verified.□

Our immediate goal is to describe a T-function in terms of stripping functions. A certain partition of X will be necessary.

DEFINITION 10. Let $E = \{x \in X\colon p\theta x = \theta p x\}$. Suppose S is a subset of X which is the disjoint union of subsets K, L, $S \cap E$, and A, and e is a function from S to the nonnegative integers such that $e(0) = 0$, and for nonzero x in S,

(1) if $x \in K$ then $px \in S$ and $e(x) = e(px) + 1$,
(2) if $x \in L$ then $\theta^{-1}(p\theta x) \in S$ and $e(x) = e(\theta^{-1}(p\theta x)) + 1$,
(3) if $x \in S \cap E$ then $px \in S$ and $e(x) = e(px) + 1$, and
(4) if $x \in A$ then $e(x) = 1$.

Under these conditions, we shall say that (S, e, K, L, A) is a *suitably partitioned subset of* X (or simply that S is suitably partitioned). If (S, e, K, L, A) and (S', e', K', L', A') are two suitably partitioned subsets of X, we shall say $S \preccurlyeq S'$ if $S \subset S'$, $K \subset K'$, $L \subset L'$, $A \subset A'$, and e' is an extension of e. Clearly, $\preccurlyeq$ is a partial order.

DEFINITION 11. If (S, e, K, L, A) is a suitably partitioned subset of X, we may construct a larger suitably partitioned subset by adjoining elements x in E for which $px \in S$, and then defining $e(x) = e(px) + 1$ for such elements. Denote by S^* the largest superset of S which can be obtained in this way, and let e^* denote the corresponding extension of e. Then (S^*, e^*, K, L, A) is a suitably partitioned subset of X.

LEMMA 2. *Let* (S, e, K, L, A) *be a suitably partitioned subset of* X, *and* F *a finite subset of* X *which is the disjoint union of sets* K_0, L_0, *and* A_0, *and for which* $F \cap (S \cup E) = \varnothing$. *Then we can construct a suitably partitioned subset* $(S', e', K', L', A') \succcurlyeq (S, e, K, L, A)$ *such that* $S \cup F \subset S'$, $K \cup K_0 = K'$, $L \cup L_0 = L'$, $A \cup A_0 \subset A'$, $S' \sim S$ *is finite, and* $S' \sim S \subset F \cup E$.

PROOF. By induction on the size of F, we may assume it consists of a single element x. If $x \in A_0$, set $e'(x) = 1$, $A' = A \cup A_0$, $K' = K$, $L' = L$, and $S' = S \cup F$. If $x \in K_0$, let m be the least positive integer such that $p^m x = 0$ or $p^m x \notin E$. If $p^m x \in S$, put $x, px, \ldots, p^{m-1}x$ in S' and define $e'(p^i x) = e(p^m x) + m - i$ for $0 \leqslant i \leqslant m$. If $p^m x \notin S$, put $p^m x$ in A', put $p^i x$ in S' for $0 \leqslant i \leqslant m$, and define $e'(p^i x) = 1 + m - i$. Define S' (and A') to include any elements we have added, in addition to S (to A). Define $K' = K \cup K_0$ and $L' = L \cup L_0$. If $x \in L_0$, let m be the least positive integer such that $\theta^{-1}(p^m \theta x) = 0$ or $\theta^{-1}(p^m\theta x) \notin E$. Proceed as for K_0, considering $\theta^{-1}(p^i\theta x)$ instead of $p^i x$ for $0 \leqslant i \leqslant m$. □

PROPOSITION 6. *Suppose* (S, e, K, L, A) *is a suitably partitioned subset of* X *with these additional properties*:

(1) *if* $x \in L \cup A$ *and* $px \neq 0$, *then there is an element* u *in* $K \cup E$ *such that* $px = pu$ *and* $hu > hx$, *and*

(2) *if* $x \in K \cup A$ *and* $p\theta x \neq 0$, *then there is an element* v *in* $L \cup E$ *such that* $p\theta x = p\theta v$ *and* $h\theta v > h\theta x$.

If $z \notin S$, *we can construct a suitably partitioned subset* $(S'', e'', K'', L'', A'') \succcurlyeq (S, e, K, L, A)$ *which contains* z *and for which* (1) *and* (2) *hold* (*with* K'', L'', A'' *replacing* K, L, A).

PROOF. We shall construct increasing sequences $\{K_i\}_{i=0}^{\infty}$, $\{L_i\}_{i=0}^{\infty}$, $\{A_i\}_{i=0}^{\infty}$ of subsets of X, such that for each i, $F_i = K_i \cup L_i \cup A_i$ is a disjoint union and $F_i \cap (S \cup E) = \varnothing$. In addition:

(1′) when i is odd and $x \in L_i \cup A_i$ with $px \neq 0$, there shall be an element u in $K_{i+1} \cup K \cup E$ so that $pu = px$ and $hu > hx$;

(2′) when i is even and $x \in K_i \cup A_i$ with $p\theta x \neq 0$, there shall be an element v in $L_{i+1} \cup L \cup E$ so that $p\theta v = p\theta x$ and $h\theta v > h\theta x$.

Finally, for each $i \geqslant 0$ we shall have a suitably partitioned subset

$$(S_i, e_i, K \cup K_i, L \cup L_i, A \cup A_i)$$
$$\preccurlyeq (S_{i+1}, e_{i+1}, K \cup K_{i+1}, L \cup L_{i+1}, A \cup A_{i+1}).$$

Since S^* satisfies (1) and (2) if S does, we shall assume $S = S^*$, and thus by induction on the exponent of z, we may assume $z \notin E$. Thus both hpz and $h(p\theta z)$ are limit ordinals or ∞.

To begin, set $F_0 = \{z\}$ and apply Lemma 2, with $e_0 = e$, $A_0 = \{z\}$, and $K_0 = L_0 = \varnothing$. Then define $S_1 = S'$, $A_1 = A' \sim A$, $K_1 = K_0$, $L_1 = L_0$, and e_1 to be the extension of e_0. Given $(S_i, e_i, K_i, L_i, A_i)$, to define F_{i+1} we proceed as follows. If i is odd, consider x in $L_i \cup A_i$ with $px \neq 0$; note then $h(px)$ is a limit ordinal. We can find an element u for which $px = pu$ and $hu > hx$, and since F_i is finite we may assume $u \notin F_i$. If $u \in S \sim (K \cup E)$ then $u \in L \cup A$ and so by (1), there is u' in $K \cup E$ with $pu' = pu$ and $hu' > hu$; thus (1′) holds. If $u \notin S \cup E$, put u in K_{i+1} so that (1′) will hold in any case. Define K_{i+1} to include all elements u added in this way, along with K_i. Define $L_{i+1} = L_i$. To define A_{i+1}, apply Lemma 2 to $F_i \cup K_{i+1}$; this gives a set A' for which $A' \sim A$ is finite; set $A_{i+1} = A_i \cup (A' \sim A)$. Now that K_{i+1}, L_{i+1}, and A_{i+1} are defined, set F_{i+1} to be their union. Apply Lemma 2 to obtain S_{i+1} and e_{i+1}.

If i is even, consider $K_i \cup A_i$ and proceed in a similar manner so that (2′) will hold. It is clear that with $S'' = \bigcup S_i$, $K'' = \bigcup K_i$, $L'' = \bigcup L_i$, $A'' = \bigcup A_i$ and e'' defined in the obvious way, the result follows. □

A Zorn's Lemma argument yields the next result.

PROPOSITION 7. *There are subsets K, L, A of X and a function e such that (X, e, K, L, A) is suitably partitioned. In addition,*

(1) if $x \in L \cup A$ with $px \neq 0$, then there is an element u in $K \cup E$ such that $px = pu$ and $hu > hx$, and

(2) if $x \in K \cup A$ with $p\theta x \neq 0$, then there is an element v in $L \cup E$ such that $p\theta x = p\theta v$ and $h\theta v > h\theta x$.

PROPOSITION 8. *The T-function θ is the composition of two strip-grafts.*

PROOF. Let e, K, L, A be such that (X, e, K, L, A) is suitably partitioned and (1) and (2) of Proposition 7 hold. For each x in X, let $x' = x$ and set $X' = \{x' : x \in X\}$, with multiplication by p on X' defined by $p(x') = (px)'$ if $x \in K \cup E$, $p(x') = 0$ if $x \in L \cup A$. With $\sigma_1 : X \to X'$ the identity function, Lemma 1 shows that X' is a p-basic tree. In view of (1), the set $\{x \in p^{-1}z: p\sigma_1(x) = \sigma_1(px)\}$ is dense for a nonzero z in X. Thus, Lemma 1 shows that σ_1 is a stripping function.

For each y in Y, let $y' = y$ and set $Y' = \{y': y \in Y\}$, with multiplication by p on Y' defined by $p(y') = (py)'$ if $\theta^{-1}y \in L \cup E$, $p(y') = 0$ if $\theta^{-1}y \in K \cup A$. Then Y' is a p-basic tree and the identity function σ_2: $Y \to Y'$ is a stripping function.

Now define $Z = \{(x, y) \in X \times Y: y = \theta x\}$, with multiplication by p in Z defined as $p(x, y) = (px, \theta(px))$ if $x \in K \cup E$; $p(x, y) = (\theta^{-1}py, py)$ if $x \in L \cup E$, and $p(x, y) = (0, 0)$ if $x \in A$. An easy induction shows that $e(x)$ is the exponent of the element (x, y) in Z, so that Z is a p-basic tree. If α is the height of (x, y) in Z, an induction on α shows that α is the common value $hx = hy$.

Let σ_3: $Z \to X'$ be defined by $\sigma_3(x, y) = x'$, and σ_4: $Z \to Y'$ be defined by $\sigma_4(x, y) = y'$. Note $p\sigma_3(x, y) = \sigma_3(p(x, y))$ for x in $K \cup E$, whereas $p\sigma_3(x, y) = 0$ for x in $L \cup A$. In view of this and (1), the set $\{(u, v) \in p^{-1}(x, y): p\sigma_3(u, v) = \sigma_3(p(u, v))\}$ is dense in $p^{-1}(x, y)$ for nonzero (x, y). Consequently, σ_3 is a stripping function; similarly, σ_4 is a stripping function. Setting $\xi_2 = \sigma_3^{-1}\sigma_1$ and $\xi_1 = \sigma_2^{-1}\sigma_4$, we observe that $\theta = \xi_1\xi_2$, which completes the proof. □

We now drop the assumption that X and Y are fully stripped. The remaining results follow easily from the preceding discussion.

THEOREM 4. *The p-basic trees X and Y are equivalent iff there is a p-basic tree Z such that:*

(1) *the natural direct sum decompositions of* $[X]$ *and* $[Z]$ *have isomorphic refinements, and*

(2) *the natural direct sum decompositions of* $[Y]$ *and* $[Z]$ *have isomorphic refinements.*

THEOREM 5 (ULM'S THEOREM FOR P-BASIC TREES). *Two p-basic trees are equivalent if and only if they have the same Ulm invariants.*

COROLLARY (ULM'S THEOREM FOR SIMPLY PRESENTED GROUPS). *Two simply presented groups are isomorphic if and only if they have the same Ulm invariants.*

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